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Ramsey Sequences with Bounded Clique Number

Abhishek Girish Aher[*] and Aparna Lakshmanan S[†]
Department of Mathematics
Cochin University of Science and Technology
Cochin - 22, Kerala, India

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Abstract

A sequence of graphs $\{G_k\}$ is a Ramsey sequence if for every positive integer k, the graph G_k is a proper subgraph of G_{k+1}, and there exists an integer $n > k$ such that every red-blue coloring of G_n contains a monochromatic copy of G_k. Among the wide range of open problems in Ramsey theory, an interesting open question is "Does there exist an ascending sequence $\{G_k\}$ with $\lim_{k\to\infty} \chi(G_k) = \infty$ and $\lim_{k\to\infty} \omega(G_k) \neq \infty$ that is a Ramsey sequence?". In this paper, we solve this problem by constructing a Ramsey sequence $\{G_k\}$ with a bounded clique number such that $\lim_{k\to\infty} \chi(G_k) = \infty$. Furthermore, using the observation that any monotonic increasing sequence of graphs that contains a Ramsey sequence as a subgraph is also Ramsey, we can generate infinitely many Ramsey sequences using this example.

Keywords: Ramsey sequence, Erdős–Hajnal shift graphs, Triangle-free graphs

AMS Subject Classification: 05C35, 05C55, 05C15

1 Introduction

One of the most prominent branches of Extremal Graph Theory is Ramsey Theory which originated from a specific case of a result by the British Philosopher, Economist and Mathematician Frank Ramsey, presented in his paper titled "On a Problem of Formal Logic" [12] published in 1930. The graph-theoretic interpretation of Ramsey's Theorem for two colors is as follows.

Theorem 1.1 *[6][Ramsey's Theorem] For any two positive integers s and t, there exists a positive integer N such that for every red-blue coloring of K_N, there is a complete subgraph K_s all of whose edges are colored red (a red K_s) or a complete subgraph K_t all of whose edges are colored blue (a blue K_t).*

[*]E-mail: abhishekaher99@gmail.com
[†]E-mail: aparnals@cusat.ac.in, aparnaren@gmail.com

The existence of classical Ramsey numbers $R(s, s)$, established indirectly by Ramsey [3], and bipartite Ramsey numbers $BR(s, s)$, introduced by Beineke and Schwenk [2] for every positive integer s, forms a cornerstone of Ramsey Theory. Chartrand and Zhang proposed a novel Ramsey concept, detailed in [4–6], involving ascending graph sequences.

A sequence of graphs $\{G_k\}$ is *ascending* if G_k is isomorphic to a proper subgraph of G_{k+1} for all positive integers k. Such a sequence is a *Ramsey sequence* if, for every k, there exists an integer $n > k$ such that every red-blue coloring of G_n yields a monochromatic G_k, either red or blue. Results by Ramsey [3] and by Beineke and Schwenk [2] demonstrate that $\{K_k\}$ and $\{K_{k,k}\}$ are Ramsey sequences. The following proposition [6] is one among the major results related to Ramsey sequences.

Proposition 1.2 ([6], Proposition 2.1) *If $\{G_k\}$ is a Ramsey sequence, then either every graph G_k is bipartite or $\lim_{k\to\infty} \chi(G_k) = \infty$.*

The converse of the above proposition is not true. For instance, in [6] it is proved that the sequence of hypercubes $\{Q_k\}$ forms an ascending sequence of bipartite graphs, but is not a Ramsey sequence and the sequence $S = \{M^k(K_3)\}$ is ascending, with $\lim_{k\to\infty} \omega(M^k(K_3)) = 3$ and $\lim_{k\to\infty} \chi(M^k(K_3)) = \infty$, but S is not a Ramsey sequence. Though the converse is not true in general, we have the following theorem which gives a subclass of ascending sequences of graphs with chromatic number tending to infinity due to the clique number tending to infinity, which turns out to be Ramsey sequences.

Theorem 1.3 ([6], Theorem 2.14) *If $\{G_k\}$ is an ascending sequence of graphs for which $\lim_{k\to\infty} \omega(G_k) = \infty$, then $\{G_k\}$ is a Ramsey sequence.*

However, there exists numerous graph sequences $\{G_k\}$ where the chromatic number tends to infinity, whereas the clique number do not. This observation prompted an open question in [5, 6]:
problem: Does there exists an ascending sequence $\{G_k\}$ with $\lim_{k\to\infty} \chi(G_k) = \infty$ and $\lim_{k\to\infty} \omega(G_k) \neq \infty$ that is a Ramsey sequence?

In this paper, we solve this open problem by providing an ascending sequence of triangle-free non-bipartite graphs - the Erdős-Hajnal shift graphs, for which the chromatic number tends to infinity and is a Ramsey sequence.

2 Erdős-Hajnal shift graphs

For integers $N \geq k \geq 2$, the *shift graph* $Sh(N, k)$ [8] is the graph whose vertices are all k-element subsets of $[N] = \{1, 2, \ldots, N\}$ and two vertices $X = \{x_1 < x_2 < \cdots < x_k\}$ and $Y = \{y_1 < y_2 < \cdots < y_k\}$ are adjacent in $Sh(N, k)$ if and only if $x_{i+1} = y_i$ for all $i = 1, 2, \ldots, k-1$. Shift graphs were further investigated in [7] and [10]. The Erdős-Hajnal shift graphs G_k, as described in lecture notes [13], is a special case of shift graphs defined as follows. For a positive integer k, the vertex set $V(G_k) = \{[i, j] \mid 1 \leq i < j \leq 2^k + 1\}$, where $[i, j]$ represents a non-degenerate closed interval with integer endpoints and two vertices $[i, j]$ and $[\ell, m]$ are adjacent if either $j = \ell$ or $m = i$. From the definition it immediately follows that the graph is triangle-free and hence $\omega(G) = 2$, where as the chromatic number $\chi(G_k) = k + 1$. This construction, originally introduced by Erdős and Hajnal [9], produces graphs with high chromatic numbers without triangles, addressing extremal properties in graph coloring.

3 Major Result

We first prove a lemma that guarantees the existence of large induced subgraphs with monochromatic outgoing edges from First Column .

Lemma 3.1 *Let G be a graph that contains an Erdős–Hajnal shift graph G_n as a subgraph, where $n = 2^{t+1}$ for some positive integer t. Then for any red-blue edge coloring of G, there exists an induced subgraph $G' \cong G_t$, such that all edges of the form $[1, j][j, k]$ in $E(G')$ are monochromatic.*

Theorem 3.2 *The sequence $\{G_k\}$ of Erdős–Hajnal shift graphs is a Ramsey sequence.*

Observation 3.3 *Let $\{G_k\}$ be an ascending sequence of graphs such that there exists a subsequence $\{G_{k_j}\}$ that is a Ramsey sequence. Then the sequence $\{G_k\}$ itself is a Ramsey sequence.*

Proof. Let $\{G_k\}$ be an ascending sequence with a subsequence $\{G_{k_j}\}$ that is a Ramsey sequence. Since $\{G_k\}$ is ascending, for every $k > 0$, there exists $k_j > 0$ such that $G_k \subseteq G_{k_j}$. As $\{G_{k_j}\}$ is a Ramsey sequence, for each k_j, there exists $k_n > 0$ such that every red-blue edge coloring of G_{k_n} has a monochromatic induced subgraph isomorphic to G_{k_j}. Since $G_k \subseteq G_{k_j}$, it follows that every red-blue edge coloring of G_{k_n} also has a monochromatic induced subgraph isomorphic to G_k. Hence, $\{G_k\}$ is a Ramsey sequence.

Corollary 3.4 *The sequence $\{G_n\} = Sh(n, 2)$ is a Ramsey sequence, for $n > 2$.*

Proof. The proof of the corollary immediately follows from Theorem 3.2 and Observation 3.3, since the sequence $\{G_k\}$ from Theorem 3.2 is a subsequence of $\{Sh(n, 2)\}$.

4 Concluding remarks

This paper is a short note that settles the open question in [5,6] about finding Ramsey sequences with a bounded clique size. Furthermore, we can generate infinite collection of such Ramsey sequences by considering ascending sequences that contain $Sh(n, 2)$ or G_k as a subsequence. Ramsey theory is a potential branch of Mathematics which demands further exploration.

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